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LETTERS TO THE EDITORS

Comment on "Equivalent conductivity of a heterogeneous medium"

IN A RECENT paper, Muralidhar [1] has investigated the thermal conductivity of a composite medium. This numerical study was based on a two-dimensional, square spatial domain. Two sides of the domain were insulated, while the other two were subject to fixed temperature boundary conditions. Effective conductivities were calculated for different patterns and numbers of inclusions in the form of circular voids. Void fractions ranged up to 20%. One of the main conclusions drawn in the paper is that 'a composite medium can be homogenized using statically determined conductivities, even for unsteady problems' [1]. It is apparent from this statement that the determination of the static equivalent conductivity is of some interest. The purpose of this comment is to re-examine the static equivalent conductivity results by comparison with classical effective medium theory.

For a two phase medium, it is well known (e.g. ref. [2]) that the effective conductivity of the medium is bounded by the volume weighted arithmetic mean conductivity, k_a, from above, and from below by the corresponding harmonic mean, k_h

$$k_a = 1 - V + V\alpha \tag{1a}$$

and

$$k_h = \left[\frac{V}{\alpha} + 1 - V\right]^{-1} \tag{1b}$$

where proportion V of the medium has conductivity α and proportion $1-V$ has a unit conductivity. Equation (1a) is the 'rule of mixtures' result given in ref. [3]. Note also that this equation corrects equation (16) in ref. [1].

Hashin and Shtrikman [4] studied effective magnetic permeabilities using a variational approach. For the case of a two phase medium they obtained improved bounds as compared with those in equation (1). Their upper (k_u) and lower (k_l) bounds are, respectively

$$k_u = 1 + \frac{V}{\dfrac{1}{\alpha-1} + \dfrac{1-V}{2}} \tag{2a}$$

and

$$k_l = \alpha + \frac{1-V}{\dfrac{1}{1-\alpha} + \dfrac{V}{2\alpha}}. \tag{2b}$$

Dagan [5] used a similar procedure in a groundwater flow application. For the latter situation, for example, it is of interest to determine the effective hydraulic conductivity of the flow domain from knowledge of the probability density function of the (assumed random) hydraulic conductivity. The following result from Dagan's work is relevent here:

$$k^* = \frac{1}{2}\left[\int \frac{f(k)\,\mathrm{d}k}{k+k^*}\right]^{-1} \tag{3}$$

where k^* is the effective conductivity of the medium, f the probability density function and the range of integration is over the domain of f. Equation (3) is based on the assumption of an unbounded, two-dimensional spatial domain over which k can vary randomly. Note that Kirkpatrick [6] provides an alternative derivation of equation (3).

For the two phase material under consideration we define f as

$$f(k) = (1-V)\delta(k-1) + V\delta(k-\alpha) \tag{4}$$

where δ is the Dirac delta function. Substituting f from equation (4) into equation (3) gives a simple quadratic for k^*

$$k^{*2} - (1-2V)(1-\alpha)k^* - \alpha = 0. \tag{5}$$

In summary, k^* gives the effective conductivity of the two phase medium with the bounds

$$k_h \leqslant k_l \leqslant k^* \leqslant k_u \leqslant k_a. \tag{6}$$

Allowing α to be non-zero corresponds to the general problem considered in ref. [1], although $\alpha \approx 0$ for the particular application of interest in the paper (cylinder blocks of IC engines). For this case equations (1), (2) and (5) give

$$k_h = k_l = \begin{cases} 1, & V = 0 \\ 0, & V > 0 \end{cases} \tag{7a,b}$$

$$k^* = \begin{cases} 1-2V, & V \leqslant 1/2 \\ 0, & V > 1/2 \end{cases} \tag{7c}$$

$$k_u = \frac{1-V}{1+V} \tag{7d}$$

and

$$k_a = 1 - V. \tag{7e}$$

Clearly, equations (7a) and (7b) are not useful. The reason for such poor quality lower bounds is, of course, that the medium is composed of material of conductivity 0 and 1, so it is possible by appropriate arrangement of the inclusions for the low-conductivity material to make the material completely insulating.

The predictions of equations (7c)–(7e) are now compared with the numerical experiments reported by Muralidhar (see Fig. 2 of ref. [1]). In Fig. 1 the numerical data are reproduced,

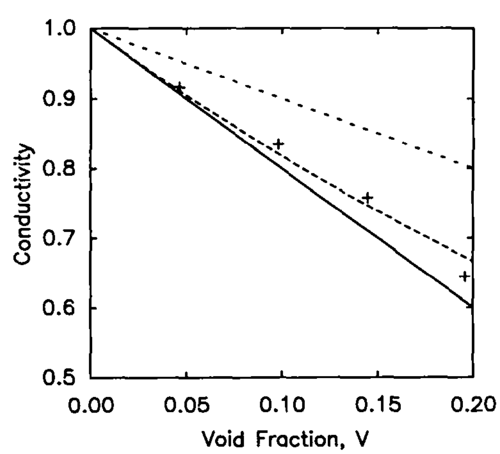

FIG. 1. Comparison of static effective conductivity vs V reported by Muralidhar [1] (pluses) with equations (7c) (solid line), (7d) (long dashes) and (7e) (short dashes).

along with curves corresponding to equations (7c)–(7e). This figure shows that the arithmetic mean, k_a, is an appropriate upper bound for the numerical data rather than k_u, whereas the prediction of k^* underestimates the results. This result is not surprising since the effect of the boundaries has been ignored. Indeed, it is reasonable to assume that the insulating side boundaries imposed in the numerical simulations will tend to increase the effective conductivity relative to a domain that is unbounded in the transverse direction. Thus, k^* acts as an approximate lower bound of the effective conductivity while k_a remains as the upper bound. These bounds may be compared with $1-1.63V$, the best straight-line fit of the numerical data [1]. It is interesting to observe that k_u gives, at first sight, a reasonable fit of the numerical data, although one may question whether the curvature is correct.

In conclusion, the finite, square domain with circular inclusions studied by Muralidhar [1] violates the assumptions of classical effective medium theory. In particular, the latter is based on the assumptions that the averaging volume is large compared with that of the inclusion, and that the number of inclusions within the medium is large. The results in Fig. 1 show, however, that the classical theory provides useful bounds on the numerical results. In the absence of other information, it appears that k^* is a reasonable estimator of the effective conductivity. Finally, the conclusion reached by Muralidhar [1] that the static effective conductivity can be used for unsteady problems corresponds with a similar conclusion for unbounded groundwater flow domains reached by Dagan [5].

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Comments on "Analysis of close-contact melting for octadecane and ice inside isothermally heated horizontal rectangular capsule"

The purpose of this letter is to point out that the thin liquid film analysis reported in ref. [1] is a special case of a more general theory published almost three years ago [2], which was apparently overlooked. Reference [2] described melting on a rectangular contact surface, with or without relative motion (sliding) between the two solid parts, and with or without heating due to viscous dissipation in the liquid film. The analysis of Hirata *et al.* qualifies as a special case of ref. [2] for three reasons. They assumed that:

(i) The rectangular contact surface is infinitely wide (the short side was labeled W in their Fig. 4).

(ii) There is no relative motion, that is, the phase-change material does not slide laterally.

(iii) Viscous dissipation in the lubricating film is negligible.

Hirata *et al.*'s key theoretical result—the film thickness formula (17)—is essentially the same as equation (21) in ref. [2]

$$\frac{h}{L}=\left[\frac{Ste}{P_n/(\alpha\mu/L^2)}\phi\right]^{1/4} \tag{21}$$

in which h is the film thickness, L the short side of the contact surface (Hirata *et al.*'s W), Ste the liquid Stefan number, α the thermal diffusivity, μ the viscosity, and $\phi = 1$ the factor accounting for the infinitely wide shape of the contact area. When the contact area is a rectangle with the same L but finite width, the factor ϕ is smaller than 1 (Fig. 2 in ref. [2]).

It is important to note that equation (21) is expressed in terms of the instantaneous average pressure (P_n) maintained between the phase-change material and the flat heater. In this way the results of ref. [2] are applicable to any geometry in which the instantaneous average pressure may change with time, for example, because of the finiteness of the solid block of phase-change material, and the size and shape of the capsule (as in Hirata *et al.*'s geometry).

Equation (21) stresses the fact that the contact melting process is *quasisteady*, i.e. decoupled from the other time-dependent features of the greater system. In this sense, the presence of time as a variable on the right-hand side of Hirata *et al.*'s film thickness formula (17) is misleading: the time-dependence entered that expression only through the instantaneous average pressure, which changes slowly with time.

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